Exam

Quantum Field Theory I

The full set of exam questions can score up to $N = 27$ points. The final grade is then computed according to the formula: Grade $= 1 + \frac{N\cdot 11}{50}$ and then rounded to the nearest quarter point, except if the resulting grade is between 3.75 and 4, in which case the final grade is systematically rounded to 4. Therefore, an exam scoring strictly more than 12.5 points will be graded at least 4, and an exam scoring strictly more than 22 points will get a 6.

Exercise 1

a) Given $[\mathscr{L}] = 4$ and $[\partial] = 1$, the dimensions of all the fields is 1, $[A_i] = [B_i] = [C_i] = 1$, while for the parameters we have $[m_i^2] = 2$ and $[\lambda] = 1$. [1 point]

b) All the Lorentz indices are suitably contracted and no explicit coordinate dependence appears in the Lagrangian: the Poincaré group is thus a symmetry.

The quadratic part of the action features invariance under independent $SO(3)$ rotations of the three fields $A_i \to R^{(A)}_{ij} A_j$, $B_i \to R^{(B)}_{ij} B_j$, $C_i \to R^{(C)}_{ij} C_j$, with $R^{(A)}$, $R^{(B)}$ and $R^{(C)}$ orthogonal 3×3 matrices with determinant 1. The quadratic part is also invariant under independent sign flips of the fields $A_i \to -A_j$, $B_i \to -B_j$, $C_i \to -C_j$, which corresponds to the group $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. Overall the quadratic part is thus invariant under $[SO(3)\times \mathbb{Z}_2]^3 = [O(3)]^3$ (Notice that $O(3) = SO(3)\times \mathbb{Z}_2$). However the last term in the Lagrangian, the cubic, is only invariant under $SO(3)$ rotations that are the same for the fields $A_i \to R_{ij}A_j$, $B_i \to R_{ij}B_j$, $C_i \to R_{ij}C_j$. This last term is invariant because,

$$\epsilon^{ijk}A_iB_jC_k \to \epsilon^{ijk}R_{il}R_{jm}R_{kn}A_lB_mC_n = \det(R)\epsilon^{lmn}A_lB_mC_n = \epsilon^{ijk}A_iB_jC_k. \tag{1}$$

Moreover also the full $\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2$ is no longer a symmetry. The only surving discrete symmetry correponds to sign flips of pairs of fields

$$\begin{aligned} &A_i \to -A_i\,, \quad B_i \to -B_i\,, \quad C_i \to C_i \\ &A_i \to -A_i\,, \quad B_i \to B_i\,, \quad C_i \to -C_i \\ &A_i \to A_i\,, \quad B_i \to -B_i\,, \quad C_i \to -C_i \end{aligned} \tag{2}$$

which corresponds to the group $\mathbb{Z}_2\times\mathbb{Z}_2$.

We thus conclude that the internal symmetry of the system is $SO(3)\times\mathbb{Z}_2\times\mathbb{Z}_2$. Let us emphasize that this is an **internal** symmetry group, which does not entail any transformation of the spacetime coordinates x^μ. It thus has nothing to do with the spacial rotation subgroup of Poincaré. [2 points]

Here, and in other points, the detailed discussion of the discrete symmetries is not required to reach full score. For pedagogical reasons we already discussed the symmetries of the quadratic terms here, rather than at point **c)** where it belongs.

c) If we set $\lambda \to 0$, the Lagrangian reduces to the quadratic part. According to the discussion above the Lagrangian is thus invariant under three independent rotations for the A, B, and C

$$A_i \to R^{(A)}_{ij}A_j, \quad B_i \to R^{(B)}_{ij}B_j, \quad C_i \to R^{(C)}_{ij}C_j. \tag{3}$$

and under the three independent $\mathbb{Z}_2$. Thus, for $\lambda = 0$ the symmetry is extended to $[O(3)]^3$. [2 points]

d) We now add all possible terms invariant under $SO(3)$ (we ignore the discrete symmetries here). We make use of the fact that $SO(3)$ has two, and only two, invariant tensors: ϵ^{ijk} and δ^{ij}.

Let's first use the Kroenecker delta to build invariants. Since all the fields have dimension 1 and they transform under the same rotation, all possible contractions of the form $\phi_i\psi_i$ with ϕ and ψ each being either A, B or C will be invariant. With this, apart from the terms that already appear in the Lagrangian, we can have mass terms with mixed fields,

$$A_iB_i, B_iC_i, A_iC_i \tag{4}$$

and also mixed kinetic terms,

$$\partial^\mu A_i \partial_\mu B_i, \partial^\mu B_i \partial_\mu C_i, \partial^\mu A_i \partial_\mu C_i. \tag{5}$$

In the same way, we can have all possible pairwise contractions of the fields (these are dimension 4 operators)

$$\begin{aligned}&(A_iA_i)(B_jB_j), (A_iA_i)(C_jC_j), (B_iB_i)(C_jC_j), (A_iA_i)^2, (B_iB_i)^2, (C_iC_i)^2, (A_iB_i)^2, (A_iC_i)^2, (B_iC_i)^2,\\ &(A_iB_i)(A_jC_j), (A_iA_i)(B_jC_j), (B_iB_i)(A_jC_j), (A_iB_i)(B_jC_j), (C_iC_i)(A_jB_j), (C_iA_i)(C_jB_j).\end{aligned} \tag{6}$$

All of these terms come with different and independent couplings.

If we imposed also the discrete symmetries, not all terms are invariant. For example the additional mixed mass terms in equation 4 are invariant only under a $\mathbb{Z}_2$ subgroup of $\mathbb{Z}_2 \times \mathbb{Z}_2$.

We can now consider terms built using the Levi-Civita tensor as well. First, there are no additional terms that can be built using one ϵ^{ijk}, as any combination with two identical fields such as

$$\epsilon^{ijk} A_i A_j B_k = -\epsilon^{jik} A_i A_j B_k = -\epsilon^{ijk} A_i A_j B_k = 0 \tag{7}$$

vanishes by the antisymmetry of the Levi-Civita tensor. One may also wonder about terms with two epsilon tensors like, $\epsilon^{ijk}\epsilon^{ilm} A_j B_k C_l B_m$. These terms are already included since,

$$\epsilon^{ijk}\epsilon^{ilm} = \delta^{jl}\delta^{km} - \delta^{jm}\delta^{kl}. \tag{8}$$

There is thus no new term with dimension less than or equal to 4 that can be built with ϵ^{ijk}. [2 points]

e) Under SO(3), $A_i \to R_{ij} A_j$, where R is a rotation matrix, or infinitesimally,

$$A_i \to A_i + \epsilon_{ijk} A_j \theta_k = A_i + \Delta^A_{ik} \theta_k, \tag{9}$$

where θ_k are small angles. The spacetime coordinates x^μ do not transform under the symmetry. Thus, the Noether current is

$$\begin{aligned} J^\mu_k &= \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_i)} \Delta^A_{ik} + \frac{\partial \mathcal{L}}{\partial(\partial_\mu B_i)} \Delta^B_{ik} + \frac{\partial \mathcal{L}}{\partial(\partial_\mu C_i)} \Delta^C_{ik} = \\ &= \epsilon_{ijk}((\partial^\mu A_i) A_j + (\partial^\mu B_i) B_j + (\partial^\mu C_i) C_j), \end{aligned} \tag{10}$$

which is conserved by the equations of motion. Now the associated charges are,

$$Q_k = \int d^3x J^0_k = \epsilon_{ijk} \int d^3x (\dot{A}_i A_j + \dot{B}_i B_j + \dot{C}_i C_j), \tag{11}$$

where $\dot{A}_i$ means the temporal derivative of A_i. [3 points]

f) The term $\Delta\mathscr{L}_1$ is not invariant under the $SO(3)$ symmetry. To see that, one can consider for example an infinitesimal transformation with parameter $\vec{\theta} = (\theta_1, 0, 0)$ as defined in equation 9. Moreover, $\mathbb{Z}_2 \times \mathbb{Z}_2$ is broken down to $\mathbb{Z}_2 : A_i \to -A_i,\ B_i \to -B_i,\ C_i \to C_i$.

However, this term is still invariant under a subgroup of $SO(3)$: $SO(2)$ transformations that act only on the components 1 and 2 of the fields (i.e rotations in the $(1,2)$ plane in field space):

$$A_i \to \mathcal{R}_{ij} A_j, \quad B_i \to \mathcal{R}_{ij} B_j, \qquad C_i \to \mathcal{R}_{ij} C_j, \qquad \text{with importantly } i, j = 1, 2 \neq 3. \tag{12}$$

where $\mathcal{R}$ is a 2×2 orthogonal matrix with determinant 1. Indeed,

$$A_1 B_2 - A_2 B_1 = \epsilon^{ij} A_i B_j \to \epsilon^{ij} \mathcal{R}_{ik} \mathcal{R}_{jl} A_k B_l = \det(\mathcal{R}) \epsilon^{ij} A_i B_j \tag{13}$$

where ϵ^{ij} is the two dimensional Levi-Civita tensor and we used that $\det(\mathcal{R}) = 1$. The rest of the Lagrangian is of course invariant under these transformations as they form a subgroup of the original $SO(3)$ symmetry.

If we add $\Delta\mathscr{L}_2$ again we break $SO(3)$ because this term is clearly not invariant under $SO(3)$ rotations and also we break again one Z_2. The $SO(2)$ transformations just described above also remain unbroken in this case as C_3 is invariant under rotations in the $(1,2)$ plane in field space. $\Delta\mathscr{L}_2$ also preserves the same residual $\mathbb{Z}_2$ as $\Delta\mathscr{L}_1 : A_i \to -A_i,\ B_i \to -B_i,\ C_i \to C_i$.

If we add both terms, the situation is the same and the $SO(3)$ symmetry is broken down to $SO(2) \times \mathbb{Z}_2$. [2 points].

Exercise 2

- Theory Question [4 points]

a) The Hamiltonian is hermitian:

$$\begin{aligned} H^\dagger &= \int d^3x\, (\psi^\dagger_\alpha (i\nabla\cdot\vec{\sigma})_{\alpha\beta}\psi_\beta)^\dagger + \frac{m}{2}(\psi^\dagger_\alpha\psi^\dagger_\beta)^\dagger\epsilon^{\alpha\beta} - \frac{m}{2}(\psi_\alpha\psi_\beta)^\dagger\epsilon^{\alpha\beta} \\ &= \int d^3x\, ((i\nabla\cdot\vec{\sigma})_{\alpha\beta}\psi_\beta)^\dagger\psi_\alpha + \frac{m}{2}\psi_\beta\psi_\alpha\epsilon^{\alpha\beta} - \frac{m}{2}\psi^\dagger_\beta\psi^\dagger_\alpha\epsilon^{\alpha\beta} \\ &= \int d^3x\, ((-i\nabla_k\psi^\dagger_\beta)(\sigma^k)_{\beta\alpha}\psi_\alpha + \frac{m}{2}\psi_\beta\psi_\alpha\epsilon^{\alpha\beta} - \frac{m}{2}\psi^\dagger_\beta\psi^\dagger_\alpha\epsilon^{\alpha\beta} \\ &= \int d^3x\, (\psi_\beta)^\dagger i\vec{\sigma}_{\beta\alpha}\cdot\nabla\psi_\alpha - \frac{m}{2}\psi_\beta\psi_\alpha\epsilon^{\beta\alpha} + \frac{m}{2}\psi^\dagger_\beta\psi^\dagger_\alpha\epsilon^{\beta\alpha} \\ &= \int d^3x\, \psi^\dagger_\alpha(i\nabla\cdot\vec{\sigma})_{\alpha\beta}\psi_\beta + \frac{m}{2}\psi^\dagger_\alpha\psi^\dagger_\beta\epsilon^{\alpha\beta} - \frac{m}{2}\psi_\alpha\psi_\beta\epsilon^{\alpha\beta} = H \end{aligned} \tag{14}$$

where in the second to third line we used the hermiticity of the Pauli matrices and also wrote explicitly the indices of the spacial derivatives. In the third to fourth line we used integration by parts in the first term. In the last line we relabelled the indices. [2 points]

b) For the time evolution equation, we use the following property: $[AB, C] = A\{B, C\} - \{A, C\}B$ to write the commutator in terms of anticommutators:

$$\begin{aligned} \dot{\psi}_\gamma(y) &\equiv i[H, \psi_\gamma(y)] \\ &= i\left[\int d^3x\, \psi^\dagger_\alpha(x)(i\nabla\cdot\vec{\sigma})_{\alpha\beta}\psi_\beta(x) + \frac{m}{2}\psi^\dagger_\alpha(x)\psi^\dagger_\beta(x)\epsilon^{\alpha\beta} - \frac{m}{2}\psi_\alpha(x)\psi_\beta(x)\epsilon^{\alpha\beta}, \psi_\gamma(y)\right] \\ &= i\int d^3x\, \Big(-\{\psi^\dagger_\alpha(x), \psi_\gamma(y)\}(i\nabla\cdot\vec{\sigma})_{\alpha\beta}\psi_\beta - \frac{m}{2}\{\psi^\dagger_\alpha(x), \psi_\gamma(y)\}\psi^\dagger_\beta\epsilon^{\alpha\beta} - \frac{m}{2}\psi^\dagger_\alpha\{\psi^\dagger_\beta(x), \psi_\gamma(y)\}\epsilon^{\alpha\beta}\Big) \\ &= (\nabla\cdot\vec{\sigma})_{\gamma\beta}\psi_\beta - im\psi^\dagger_\beta\epsilon^{\gamma\beta} \end{aligned} \tag{15}$$

Or rearranging the terms

$$i(\bar{\sigma}^\mu\partial_\mu\psi)_\alpha = m\epsilon_{\alpha\beta}\psi^\dagger_\beta. \tag{16}$$

[3 points]

c) Recall that

$$d\Omega_{\mathbf{p}} = \frac{d^3\mathbf{p}}{(2\pi)^3 2\omega_{\mathbf{p}}} \tag{17}$$

and under the transformation $\mathbf{p} \to \mathbf{p}' = -\mathbf{p}$, we have that $\omega_{\mathbf{p}'} = \omega_{\mathbf{p}}$ which leaves the measur invariant: $d\Omega_{\mathbf{p}'} = d\Omega_{\mathbf{p}}$. So, overall

$$\int d\Omega_{\mathbf{p}}\, f(\mathbf{p}) = \int d\Omega_{\mathbf{p}'}\, f(-\mathbf{p}') \tag{18}$$

The expression for $\psi(t, \mathbf{x})$ can be divided into two terms

$$\psi(t, \mathbf{x}) = \int d\Omega_{\mathbf{p}}\left(e^{i\mathbf{p}\mathbf{x} - i\omega_{\mathbf{p}}t}\xi_+(\mathbf{p}) + e^{i\mathbf{p}\mathbf{x} + i\omega_{\mathbf{p}}t}\xi_-(\mathbf{p})\right) = \int d\Omega_{\mathbf{p}}\left(e^{-ip\cdot x}\xi_+(\mathbf{p}) + e^{i\mathbf{p}\mathbf{x} + i\omega_{\mathbf{p}}t}\xi_-(\mathbf{p})\right) \tag{19}$$

and the first term is already in the form we want. Then, by performing the change of integration variable $\mathbf{p}' = -\mathbf{p}$ on the second term it reads

$$\int d\Omega_{\mathbf{p}'}\left(e^{-i\mathbf{p}'\mathbf{x} + i\omega_{\mathbf{p}'}t}\xi_-(-\mathbf{p}')\right) = \int d\Omega_{\mathbf{p}}\left(e^{-i\mathbf{p}\mathbf{x} + i\omega_{\mathbf{p}}t}\xi_-(-\mathbf{p})\right) = \int d\Omega_{\mathbf{p}}\left(e^{ip\cdot x}\xi_-(-\mathbf{p})\right) \tag{20}$$

So, putting everything together

$$\psi(t, \mathbf{x}) = \int d\Omega_{\mathbf{p}}\left(e^{-ip\cdot x}\xi_+(\mathbf{p}) + e^{ip\cdot x}\xi_-(-\mathbf{p})\right) \tag{21}$$

[1 point]

d) In all that follows $\xi^\dagger_\alpha \equiv \xi^*_\alpha$.

From eq. 21 we have

$$i\bar\sigma^\mu \partial_\mu \psi(t,\mathbf{x}) = \bar\sigma^\mu \int d\Omega_{\mathbf{p}} (p_\mu e^{-ip\cdot x}\xi_+(\mathbf{p}) - p_\mu e^{ip\cdot x}\xi_-(-\mathbf{p})). \tag{22}$$

One also easily sees that

$$m\epsilon_{\alpha\beta}\psi^\dagger_\beta(t,\mathbf{x}) = \int d\Omega_{\mathbf{p}} \left(e^{ip\cdot x} m\epsilon_{\alpha\beta}\xi^\dagger_{+,\beta}(\mathbf{p}) + e^{-ip\cdot x} m\epsilon_{\alpha\beta}\xi^\dagger_{-,\beta}(-\mathbf{p})\right) \tag{23}$$

By equating the terms proportional to $e^{-ip\cdot x}$ in equation 22 and 23 we thus get

$$\bar\sigma^\mu p_\mu \xi_+(\mathbf{p}) = m\epsilon\xi^\dagger_-(-\mathbf{p}), \tag{24}$$

Doing the same thing for the terms proportional to $e^{ip\cdot x}$, we get a second equation

$$-p_\mu\bar\sigma^\mu \xi_-(-\mathbf{p}) = m\epsilon\xi^\dagger_+(\mathbf{p}). \tag{25}$$

To get it in the form asked in the question, we take the complex conjugate of this equation:

$$p_\mu(\bar\sigma^\mu)^*_{\alpha\beta}\epsilon_{\beta\gamma}\epsilon_{\gamma\delta}\xi^\dagger_{-,\delta}(-\mathbf{p}) = m\epsilon_{\alpha\beta}\xi_{+,\beta}(\mathbf{p}). \tag{26}$$

where we introduced $\epsilon^2 = -1$ on the left hand side. Multiplying on the left by ϵ and using

$$\epsilon^{-1}(\bar\sigma^\mu)^*\epsilon = -\epsilon(\bar\sigma^\mu)^*\epsilon = \sigma^\mu, \tag{27}$$

we get

$$\sigma^\mu p_\mu \epsilon\xi^*_-(-\mathbf{p}) = m\xi_+(\mathbf{p}). \tag{28}$$

[2 points]

e) By eqs.24 and 28 the 4-spinor Ξ satisfies the Dirac equation $(\not p - m)\Xi = 0$ since

$$\gamma^\mu p_\mu \Xi - m\Xi = \begin{pmatrix} 0 & \sigma^\mu p_\mu \\ \bar\sigma^\mu p_\mu & 0\end{pmatrix}\begin{pmatrix}\xi_+(\mathbf{p}) \\ \epsilon\xi^*_-(-\mathbf{p})\end{pmatrix} - m\begin{pmatrix}\xi_+(\mathbf{p}) \\ \epsilon\xi^*_-(-\mathbf{p})\end{pmatrix} = \begin{pmatrix}\sigma^\mu p_\mu\epsilon\xi^*_-(-\mathbf{p}) \\ \bar\sigma^\mu p_\mu\xi_+(\mathbf{p})\end{pmatrix} - m\begin{pmatrix}\xi_+(\mathbf{p}) \\ \epsilon\xi^*_-(-\mathbf{p})\end{pmatrix} = 0 \tag{29}$$

[1 points]

f) Recall that the general solution of $(\not p - m)u = 0$ has the form

$$u = \begin{pmatrix}\sqrt{p\cdot\sigma}\,\xi(\mathbf{p}) \\ \sqrt{p\cdot\bar\sigma}\,\xi(\mathbf{p})\end{pmatrix} \tag{30}$$

with $\xi(\mathbf{p})$ a generic bispinor. Equating the above form of the general solution with the expression for Ξ in eq. 29 we then have

$$\xi_+(\mathbf{p}) = \sqrt{p\cdot\sigma}\,\xi(\mathbf{p}), \qquad \epsilon\xi^*_-(-\mathbf{p}) = \sqrt{p\cdot\bar\sigma}\,\xi(\mathbf{p})\,. \tag{31}$$

The first of these equations already proves one of the identities. For the second, using $\epsilon^{-1}\sigma_i\epsilon = -\sigma^*_i$ and the explicit expressions for $p\cdot\bar\sigma$ and $p\cdot\sigma$

$$\sqrt{p\cdot\bar\sigma} = \frac{1}{2}\left(\sqrt{\omega_{\mathbf{p}}+m} + \frac{\mathbf{p}\cdot\sigma}{\sqrt{\omega_{\mathbf{p}}+m}}\right), \quad \sqrt{p\cdot\sigma} = \frac{1}{2}\left(\sqrt{\omega_{\mathbf{p}}+m} - \frac{\mathbf{p}\cdot\sigma}{\sqrt{\omega_{\mathbf{p}}+m}}\right) \tag{32}$$

we get

$$\begin{aligned}\xi^*_-(-\mathbf{p}) &= \epsilon^{-1}\sqrt{p\cdot\bar\sigma}\,\xi(\mathbf{p}) = -\epsilon^{-1}\sqrt{p\cdot\bar\sigma}\,\epsilon^2\xi(\mathbf{p}) = -\frac{1}{2}\left(\sqrt{\omega_{\mathbf{p}}+m}\;\epsilon^{-1}\epsilon + \frac{p_i\epsilon^{-1}\sigma_i\epsilon}{\sqrt{\omega_{\mathbf{p}}+m}}\right)\epsilon\xi(\mathbf{p}) \\ &= -\frac{1}{2}\left(\sqrt{\omega_{\mathbf{p}}+m}\; - \frac{p_i\sigma^*_i}{\sqrt{\omega_{\mathbf{p}}+m}}\right)\epsilon\xi(\mathbf{p})\end{aligned} \tag{33}$$

and taking the complex conjugate

$$\xi_-(-\mathbf{p}) = -\frac{1}{2}\left(\sqrt{\omega_{\mathbf{p}}+m}\; - \frac{p_i\sigma_i}{\sqrt{\omega_{\mathbf{p}}+m}}\right)\epsilon^*\xi^*(\mathbf{p}) = -\sqrt{p\cdot\sigma}\,\epsilon\xi^*(\mathbf{p}) \tag{34}$$

[2 points]